Math 291

An overview of the exam problems.
Take a minute to look at all the questions, THEN solve each problem on its corresponding page INSIDE the booklet.

1. Let the function $f(x)$ be given by

$$f(x) = \begin{cases} 0, & \text{when } 0 \le x < \pi \\ x - \pi, & \text{when } \pi \le x < 2\pi \\ \text{and } f(x) \text{ is periodic with period } 2\pi. & \end{cases}$$

a) (5 pts) Sketch the graph of $f(x)$ for $x \in [-2\pi, 4\pi]$.
b) (10 pts) The Fourier series of $f(x)$ is $\sum_{n\ge 0} a_n \cos nx + \sum_{n\ge 1} b_n \sin nx$. Find ONLY the coefficients b_n.

2. a) (6 pts) Plot the polar graph of the curve $C : r = 1 + \sin\theta$. Also draw the line $L : y = 4/9$ on your graph.
b) (3 pts) Convert the equation of L to polar coordinates.
c) (6 pts) Find the (r, θ)-coordinates of the two points of intersection on $L \cap C$.

3. Consider the following moving point in space:

$$P(t) = (3t, \sqrt{6}\, e^t, \frac{1}{2}e^{2t}).$$

a) (5 pts) Find the velocity and the speed of $P(t)$ at the instant $t = 0$.
b) (5 pts) What is the arclength of the curve given by $P(t)$ for $0 \le t \le \ln 5$? Simplify your answer.
c) (5 pts) Suppose we have a function $f(x, y, z)$ with the property

$$\vec{\nabla} f|_{(3,\sqrt{6}\,e,\frac{1}{2}e^2)} = (e^2, -\sqrt{6}\,e, 5).$$

Find $\dfrac{d}{dt} f(P(t))$ at the instant when the point $P(t)$ passes through $(3, \sqrt{6}\,e, \frac{1}{2}e^2)$.

4. a) (8 pts) Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^4 + y^2}$ does not exist.
b) (8 pts) Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^2 + y^2}$ does exist (hint: the limit is equal to 0).

5. Consider the function $f(x, y, z) = ze^{x^3 y}$.
a) (6 pts) Find the gradient of $f(x, y, z)$ at $P_0(1, 1, -1)$.
b) (7 pts) Find the equation of the tangent plane to the surface $f(x, y, z) = -e$ at P_0.
c) (6 pts) Determine the direction in which $f(x, y, z)$ increases most rapidly when the point (x, y, z) moves away from P_0. Your answer should be a **unit** vector.

6. Given a function $f(x, y)$ satisfying $f(1, 2) = 4$, $\vec{\nabla} f\,\Big|_{(1,2)} = (3, 4)$.
a) (6 pts) Approximately how much is $f(1.03, 1.99)$?
b) (7 pts) Find a direction $\vec{u}$ in which the directional derivative $D_{\vec{u}} f\,\Big|_{(1,2)} = 0$. Your answer $\vec{u}$ should be a **unit** vector.
c) (7 pts) Let S be the graph of f. In other words, $S = \{(x, y, z) \in \mathbf{R}^3 \mid z = f(x, y)\}$. Find the equation of the tangent plane to S at the point $P_0(1, 2, 4) \in S$. (Be careful.)